Granular flow in a three-dimensional rotating container

T. Elperin* and A. Vikhansky†
The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel
(Received 2 November 1999)

We considered a flow of a cohesionless granular material in a partially filled three-dimensional rotating container. A model is suggested to describe a density of the surface flow and the profile of the free surface of the granular material. It is shown that when the container has an ellipsoidal shape the obtained system of partial differential equations can be reduced to a set of two ordinary differential equations.

PACS number(s): 83.70.Fn, 45.05.+x, 05.60.−k

Dynamics, mixing, and separation of granular materials in partially filled containers have been the subject of numerous experimental and theoretical investigations (see, e.g., [1–6]). Although different aspects of granular dynamics in two-dimensional drums were studied analytically, there were no attempts to describe a continuous granular flow in a three-dimensional rotating container. Note, that while in a two-dimensional slowly rotating drum the free surface of a granular material has a nearly flat profile with a nearly constant angle of inclination, in a three-dimensional case the situation changes drastically. The free surface evolves with time in a nontrivial manner, which depends on the shape of the drum, and also on the way the mixer is tumbled. In a two-dimensional case, the axis of rotation is always directed normally to the drum while in a three-dimensional case it can be, for example, wobbled, which causes an additional axial flow of granular material [2]. Thus, determining the granular flow even in the most simple case of a constant angle of repose of granular material requires the solution of a system of partial differential equations of sandheap evolution [7,8]. In the present work, we considered granular flow in an ellipsoidal mixer. Due to the simple analytical form of the container, the system of partial differential equations can be reduced to a system of two ordinary differential equations.

Let us consider an ellipsoidal drum with semiaxes A, B, C (Fig. 1) that rotates with a constant angular velocity ω around the y axis. At the initial moment $t=0$ the container is inclined with respect to a laboratory frame of reference by the angles ψ (angle of precession) and ϕ (angle of nutation). The drum is partially filled with a granular material with a constant bulk density ρ and a constant angle of repose μ. We assume that the angular velocity ω is sufficiently large to cause continuous avalanches while the inertial forces are much smaller than the gravity and friction forces, e.g., Froude number $Fr^2 \equiv \omega^2 L/g \ll 1$, where L is a characteristic size of the drum. Thus the only characteristic time of the process is $1/\omega$, and for simplicity we set the angular velocity to be 1.

The flow of the granular material in the drum can be described as follows. Particles rotate with the bulk of the granular material and fall down into a thin cascading layer when they reach the free surface $z=h(t,x,y)$, which has a constant angle of inclination with respect to the horizontal:

$$(\nabla h)^2 = \tan^2(\mu) = \gamma^2. \tag{1}$$

Certainly, the latter assumption is an approximation whereby Eq. (1) can be viewed as a zero-order term in the expansion of the momentum conservation equation in Froude number series (see, e.g., [1]). In the present study, we adopted the model of sandpile evolution described in [7,8]. Recently this model was used to explain the phenomenon of formation of transversal bands in slowly rotating cylindrical containers filled with binary granular mixtures [9]. The brief description of this model is presented in the following.

Assume that the flow of the granular material occurs only in a very thin boundary layer and it does not involve the stationary bulk of the material. Denote the horizontal projection of the mass flux density per unit area by $\rho\overline{q}(t,x,y)$. Since inertial forces are small, the material flux is directed toward the steepest descent of the free surface:

$$\rho\overline{q} = -k\rho\vec{\nabla} h, \tag{2}$$

where the flow rate $k(t,x,y) \geqslant 0$ is the unknown scalar function. Thus, the equation of mass balance for granular material reads

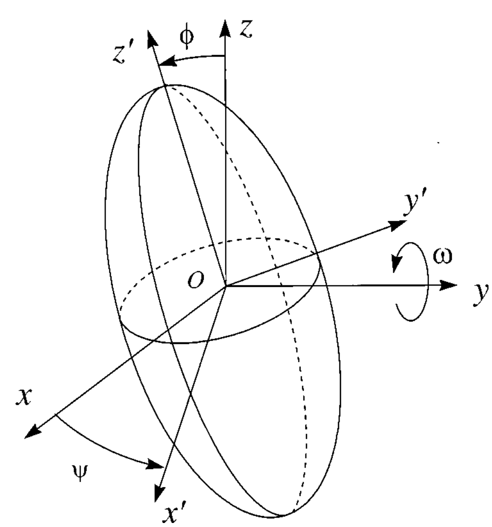

FIG. 1. Schematic view of the flow of granular material in a rotating ellipsoidal drum and the coordinates system.

*Email address: elperin@menix.bgu.ac.il
†Email address: vikhal@menix.bgu.ac.il

$$\frac{\partial h}{\partial t}+\vec{u}\cdot\vec{\nabla}h-w=\vec{\nabla}\cdot(k\vec{\nabla}h), \tag{3}$$

where $\vec{u}=(h,0)$ and $w=-x$ are horizontal and vertical components of the rotationally induced bulk velocity at the free surface.

A horizontal projection of an intersection of the free surface with the walls of the container is an unknown closed curve Γ on the (x, y) plane. Since the walls are impermeable, the boundary condition for Eq. (3) reads

$$q_n|_\Gamma=0. \tag{4}$$

Equations (1), (3), and (4) with respect to the unknown functions $k(t,x,y)$ and $h(t,x,y)$ provide a closed mathematical formulation of the problem.

Note that the direct solution of the above equations for an arbitrary rotating container is a very complicated problem. Even in the more simple case of evolution of a sandpile growing on an arbitrary rigid support surface due to an external source of granular material, the problem requires a variational formulation, and only a numerical solution is feasible (for details see [7,8]). The only case that was investigated analytically is growth and interaction of conical piles on a flat support surface [7,10–12].

Fortunately, in an ellipsoidal drum the solution procedure can be greatly simplified. Let us assume that the free surface is flat. Then, the unit vector normal to the free surface can be written in spherical coordinates α, μ as follows

$$\vec{n}=(\sin(\mu)\cos(\alpha),\sin(\mu)\sin(\alpha),\cos(\mu)),$$

where $\alpha(t)$ is angle of orientation of the free surface with respect to the x axis. Thus, the equation of the free surface reads

$$h(t,x,y)=-\gamma\{x\cos[\alpha(t)]+y\sin[\alpha(t)]\}+h_0(t), \tag{5}$$

where h_0 is the height of the free surface at the origin of the coordinates. The horizontal projection of the intersection between the free surface and the walls of the container is an ellipse

$$a(t)(x-x_0)^2+2b(t)(x-x_0)(y-y_0)+c(t)(y-y_0)^2=1.$$

Substituting Eq. (5) into Eq. (3), we find that the left-hand side of Eq. (3) is a linear function of x and y. Therefore the right-hand side of Eq. (3), $\vec{\nabla}\cdot(k\vec{\nabla}h)$, is also a linear function of x and y. Taking into account the equation for the free surface $h(t,x,y)$ in Eq. (5), we conclude that the flow rate $k(t,x,y)$ is a quadratic function of x and y. In order to satisfy the boundary condition (4) we must assume that the flow rate k in Eq. (2) has the following form:

$$k(t,x,y)=\kappa(t)[1-a(x-x_0)^2 -2b(x-x_0)(y-y_0)-c(y-y_0)^2]. \tag{6}$$

Substituting Eqs. (5) and (6) into Eq. (3) and equating the terms of the same order with respect to x and y we obtain

$$1:\quad \frac{dh_0}{dt}=\gamma\cos(\alpha)h_0-2\gamma\kappa[(ax_0+by_0)\cos(\alpha) +(bx_0+cy_0)\sin(\alpha)], \tag{7}$$

$$x:\quad \gamma\sin(\alpha)\frac{d\alpha}{dt}+\gamma^2\cos^2(\alpha)+1 =2\gamma\kappa[a\cos(\alpha)+b\sin(\alpha)],$$

$$y:\quad -\gamma\cos(\alpha)\frac{d\alpha}{dt}+\gamma^2\cos(\alpha)\sin(\alpha) =2\gamma\kappa[b\cos(\alpha)+c\sin(\alpha)].$$

Combining the latter two equations yields

$$\frac{d\alpha}{dt}=-\frac{\sin(\alpha)}{\gamma}+2\kappa[(a-c)\sin(\alpha)\cos(\alpha) +b\{\sin^2(\alpha)-\cos^2(\alpha)\}], \tag{8}$$

$$\kappa=\frac{1}{2}(\gamma+\gamma^{-1}\cos(\alpha)[a\cos^2(\alpha) +2b\sin(\alpha)\cos(\alpha)+c\sin^2(\alpha)]^{-1}. \tag{9}$$

The parameters a, b, c, x_0, y_0 can be evaluated as follows. In the laboratory frame of reference the equation of the ellipsoid reads

$$f(t,x,y,z)=(x,y,z)^T\cdot\mathbf{F}(t,\psi,\phi)\cdot(x,y,z)=1,$$

where the time-dependent matrix

$$\mathbf{F}=(\mathbf{A}_2\mathbf{A}_1\mathbf{A}_3)^T\mathbf{F}_0(\mathbf{A}_2\mathbf{A}_1\mathbf{A}_3)$$

and

$$\mathbf{A}_1=\begin{pmatrix}\cos(\psi) & \sin(\psi) & 0\\ -\sin(\psi) & \cos(\psi) & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\mathbf{A}_2=\begin{pmatrix}0 & 0 & 1\\ \cos(\phi) & \sin(\phi) & 0\\ -\sin(\phi) & \cos(\phi) & 0\end{pmatrix},$$

$$\mathbf{A}_3=\begin{pmatrix}\cos(t) & 0 & -\sin(t)\\ 0 & 1 & 0\\ \sin(t) & 0 & \cos(t)\end{pmatrix},$$

$$\mathbf{F}_0=\begin{pmatrix}A^{-2} & 0 & 0\\ 0 & B^{-2} & 0\\ 0 & 0 & C^{-2}\end{pmatrix}.$$

Thus, the equations of the horizontal projection of the intersection between the free surface and the walls of the container read

$$a(x-x_0)^2+2b(x-x_0)(y-y_0)+c(y-y_0)^2 =f(t,x,y,h(t,x,y))=\hat{f}(t,x,y).$$

Differentiating the latter equation yields

$$a=\frac{1}{2}\frac{\partial^2\hat{f}}{\partial x^2}=(1,0,-\gamma\cos(\alpha))^T\cdot\mathbf{F}\cdot(1,0,-\gamma\cos(\alpha)), \tag{10}$$

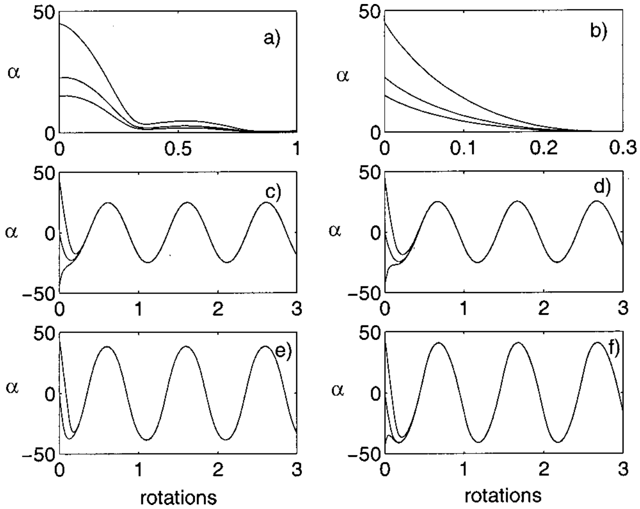

FIG. 2. Time evolution of the angle of orientation of the free surface α (in degrees): (a) $A=1$, $B=3$, $C=3$, $\psi=0$, $\phi=0$; (b) $A=1$, $B=1$, $C=3$, $\psi=0$, $\phi=0$; (c) $A=1$, $B=1.6$, $C=1.6$, $\psi=\pi/4$, $\phi=0$; (d) $A=1$, $B=1.6$, $C=1$, $\psi=\pi/4$, $\phi=0$; (e) $A=1$, $B=2.4$, $C=2.4$, $\psi=\pi/4$, $\phi=0$; (e) $A=1$, $B=2.4$, $C=1$, $\psi=\pi/4$, $\phi=0$.

$$b=\frac{1}{2}\frac{\partial^2\hat{f}}{\partial x\partial y}=(1,0,-\gamma\cos(\alpha))^T\cdot\mathbf{F}\cdot(0,1,-\gamma\sin(\alpha)), \tag{11}$$

$$c=\frac{1}{2}\frac{\partial^2\hat{f}}{\partial x^2}=(0,1,-\gamma\sin(\alpha))^T\cdot\mathbf{F}\cdot(0,1,-\gamma\sin(\alpha)), \tag{12}$$

$$(ax_0+by_0)=-\frac{1}{2}\frac{\partial\hat{f}}{\partial x}\bigg|_{x=0,y=0}=-(1,0,-\gamma\cos(\alpha))^T\cdot\mathbf{F}\cdot(0,0,h_0), \tag{13}$$

$$(bx_0+cy_0)=-\frac{1}{2}\frac{\partial\hat{f}}{\partial y}\bigg|_{x=0,y=0}=-(0,1,-\gamma\sin(\alpha))^T\cdot\mathbf{F}\cdot(0,0,h_0). \tag{14}$$

Thus, the ordinary differential equations (7) and (8) together with relations (9)–(14) provide a solution of the system of partial differential equations (1)–(4). Since the solution of the variational problem describing sandpile evolution is unique (for details see [13]), the above solution is a general solution of the system of partial differential equations (1)–(4) for an ellipsoidal mixer.

Note, that time dependence of the angle of orientation of the free surface α is independent on its height at the coordinate origin h_0. Thus the evolution of α is independent on the filling level of the container. In Figs. 2 and 3, we showed the results of the numerical solution of Eqs. (7) and (8). The angle α reaches a steady state very rapidly, irrespectively to its initial value. It was found that for containers with arbitrary (but nonequal) values of semiaxes *A, B, C,* which are initially inclined with respect to the axis of rotation, the angle α has a qualitatively similar behavior. The only parameters, which determine the amplitude of the free-surface rotation, are the largest-to-smallest axes ratio and the angle of the container with respect to the rotation axis. Figure 3 shows contour plots of the projection of the flow rate on the horizontal plane at different times. Since the free surface of granular material is a flat plane, the vectors of flux are parallel.

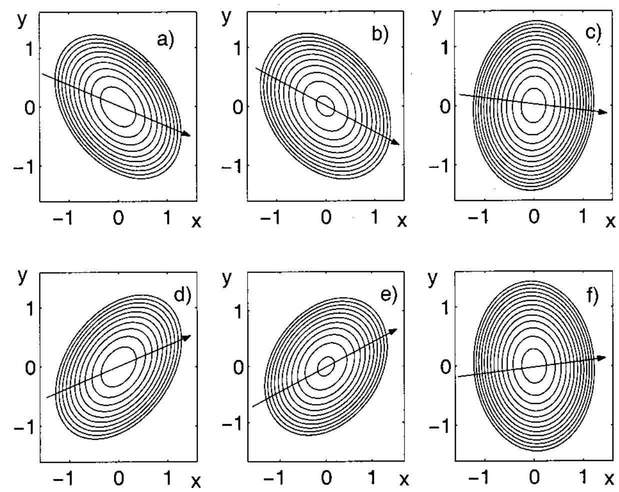

FIG. 3. Contour plots of constant flow rate in a half-filled drum for $A=1$, $B=1.6$, $C=1.6$, $\psi=\pi/4$, $\phi=0$ at different times. Zero level corresponds to the boundary of a drum. The increment between adjacent contour lines is 0.1. Arrows indicate direction of the flow of the granular material that avalanches down the free surface: (a) $t=0$, maximum flow rate=0.98; (b) $t=\pi/3$, maximum flow rate=1.02; (c) $t=2\pi/3$, maximum flow rate=1.25; (d) $t=\pi$, maximum flow rate=0.98; (e) $t=4\pi/3$, maximum flow rate=1.02; (f) $t=5\pi/3$, maximum flow rate=1.25.

It was noted in Refs. [2,6] that one of the disadvantages of the tumbling mixers is a poor material mixing in the axial direction. When a container is rotated around its nonprincipal axis, the orientation of the free surface and, therefore, the granular flux direction change periodically, which causes an enhanced transport of the granular material along the axis of rotation. Thus, in order to enhance mixing, one can rotate the drum with respect to a nonprincipal axis. Note, that other possibilities to enhance mixing, e.g., periodical wobbling of the rotation axis, also can be described by the above model after only small modifications.

In summary, we have analyzed granular flow in a three-dimensional ellipsoidal rotating drum. It was found that inclination of the mixer with respect to the plane of rotation is sufficient to enhance transport of the granular material in the axial direction. Note that the flow of granular material in the ellipsoidal container can serve as a model of a granular flow in a double-cone mixer, which is one of the most commonly used mixers used in industry [6]. The system of partial differential equations for the free-surface profile and surface flow rate is reduced to a set of two ordinary differential equations.

The work of A.V. was supported by Levy Eshcol Foundation administered by Israel Ministry of Science.

[1] A. A. Boateng and P. V. Barr, Chem. Eng. Sci. **17**, 4167 (1996).
[2] J. J. McCarthy, T. Shinbrot, G. Metcalfe, J. E. Wolf, and J. M. Ottino, AIChE J. **42**, 3351 (1996).
[3] D. V. Khakhar, T. Shinbrot, J. J. McCarthy, and J. M. Ottino, Phys. Fluids **9**, 31 (1997).
[4] T. Elperin and A. Vikhansky, Europhys. Lett. **42**, 619 (1998); **43**, 17 (1998); Chaos **9**, 910 (1999).
[5] D. V. Khakhar, J. J. McCarthy, J. F. Gilchrist, and J. M. Ottino, Chaos **9**, 195 (1999).
[6] A. W. Chester, J. A. Kowalski, M. E. Coles, E. L. Muegge, F. J. Muzzio, and D. Brone, Powder Technol. **102**, 85 (1999).
[7] L. Prigozhin, Chem. Eng. Sci. **48**, 3647 (1993); Phys. Rev. E **49**, 1161 (1994).
[8] T. Elperin and A. Vikhansky, Phys. Rev. E **53**, 4536 (1996); **55**, 5785 (1997).
[9] T. Elperin and A. Vikhansky, Phys. Rev. E **60**, 1946 (1999).
[10] G. Aronsson, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **22**, 437 (1972).
[11] G. Aronsson, L. C. Evans, and Y. Wu, J. Diff. Eqns. **131**, 304 (1996).
[12] L. C. Evans, M. Feldman, and R. F. Gariepy, J. Diff. Eqns. **137**, 166 (1997).
[13] L. Prigozhin, European J. Appl. Math. **7**, 225 (1996).